Resonant Excitation of High Amplitude Oscillations and Hydrodynamic Wave Breaking in a Streaming Cold Plasma

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The excitation of large amplitude electron oscillations in a streaming cold plasma and the minimum threshold of wave breaking in the resonant region are investigated analytically as a function of flow velocity. The problem is reduced to the solution of a driven harmonic oscillator with time varying eigenfrequency $\omega_p(t)$ in a self-consistent, stationary ion density profile. An analytical solution is presented and applied to the correct wave breaking criterion in a streaming plasma. Wave breaking sets in when the driver amplitude $\hat{E}_d$ obeys the inequality

$$\hat{E}_d > \frac{m_e \omega v_c}{e \left| \frac{1}{2} + i \eta e^{i\eta^2} \int\limits_{-\infty}^{\eta} e^{-i\eta^2} \mathrm{d}\eta, \right|}$$

which shows that the threshold is proportional to the driver frequency ω and to the flow velocity at the resonance point, v_c; however, it is independent of the density scale length. Resonance ends at $\eta = \pi/2$. The denominator assumes there the value 2.759. η is a dimensionless time which measures the transit time of a volume element through resonance.

1. Introduction

In the critical region of an inhomogeneous plasma the incident electromagnetic wave can resonantly couple to an electrostatic wave. The classical treatment of this resonance absorption has been given in Ref. [1] in a linearized form for a plasma at rest. The most complete treatment can be found in Ref. 11. A simplified nonlinear description of this process for a cold plasma is feasible if the simplification of the so called capacitor model is introduced: The sinusoidal driver field $E_d e^{-i\omega t}$ is assumed to have only a component in the direction of the density gradient. In this case the oscillation amplitude $x_e(x, t)$ of the electrons is governed by the linear equation (2)

$$\frac{\mathrm{d}^2 x_e}{\mathrm{d}t^2} + \omega_p^2(x)\, x_e = -\frac{e}{m_e} E_d\, e^{-i\omega t}, \qquad (1)$$

which is a good approximation even for large amplitudes x_e because the plasma frequency $\omega_p = (Z e^2 n_0/\varepsilon_0 m_e)^{1/2}$ only depends on the ion density $n_0(x)$. At the position $\omega_p = \omega$ the solution of (1) is given by

$$x_e(x, t) = -i \frac{e}{2 m_e \omega} t E_d\, e^{-i\omega t}, \qquad (2)$$

which shows that the amplitude grows linearly in time. As a consequence wave breaking occurs [3]. In order to distinguish this braking mechanism from particle kinetic effects [4], we call it hydrodynamic wave breaking. It is defined as the interpenetration of two volume elements of the fluid under consideration. The mathematical criterion for hydrodynamic breaking in a homogeneous plasma with the ions at rest is [5]

$$\partial x_e / \partial x < -1 \qquad (3)$$

in at least one point x.

According to (1) hydrodynamic wave breaking occurs at arbitrarily small driver fields. However, there are several damping mechanisms which introduce a finite breaking threshold for E_d. Collisions between electrons and ions can limit the indefinite growth of x_e, but in laser produced plasmas the collision frequency is generally too low to be effective. Another more effective threshold is repre-

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sented by temperature effects. At the moment of breaking the electron density becomes infinite in accordance with (3) and the thermal pressure strongly reacts. Or, in other terms, a finite temperature forces the electrostatic oscillations to propagate down the density gradient and to lead to a continuous flow of energy but of the resonance region, which may limit the oscillation amplitude x_e to a finite value. This effect was studied in numerical simulations as well as in simple analytical models [6]. Other limitations on the indefinite growth of x_e may be imposed by Landau damping [7], i.e. particle acceleration, and by distortions of the ion density profile n_0 due to light pressure. However, no analytical treatment of these effects is available as yet. The same holds for plasma convection. Many plasmas show considerable flow velocity at the critical point ($\omega_p = \omega$), e.g. laser produced plasmas. This has two effects on wave breaking. Firstly, a volume element stays for only a restricted time in the resonance region and, secondly, owing to the rarefaction of the ion density, steepening of the electron wave is reduced.

In the following we present an analytical treatment of the problem of wave breaking in a streaming cold plasma. In Section 2 we formulate the problem in terms of a driven harmonic oscillator with time varying eigenfrequency. In Section 3 a self-consistent, stationary density profile is determined and the oscillator equation is solved analytically for it. Finally, in Section 4 the correct wave breaking criterion for a streaming plasma is applied and the minimum threshold for wave breaking is derived. The calculated values of electromagnetic wave intensities show that plasma convection leads to appreciable intensities at which hydrodynamic breaking sets in. However, it also appears that in laser produced plasmas temperature effects may represent a more effective damping mechanism.

2. Resonant Excitation of Electrostatic Oscillations in a Streaming Plasma

We consider a layered medium, i.e. a plasma with its density gradient in the x-direction only. The plasma is fully ionized, the electrons are assumed to follow an adiabatic equation of state, the ion density and flow velocity are n_0 and v_0, respectively, and the electrons with density $n_e(x, t)$ are assumed to move at speed $u(x, t)$. The obliquely incident p-polarized electromagnetic wave resonantly interacts with the electrostatic wave in a small region around the critical point. Since the magnetic field amplitude has a local maximum there and since the variation of $\boldsymbol{E}$ at that point is mainly in the direction of the gradient, the following approximations are justified:

$$\nabla \times \boldsymbol{B} = (i k_y B, -\partial B/\partial x, 0) \cong (i k_y B, 0, 0), \quad (4)$$

$$\nabla \boldsymbol{E} = \partial E_x/\partial x + i k_y E_y \cong \partial E_x/\partial x, \quad (5)$$

where k_y is the wave vector component in the y-direction, $k_y = k_0 \sin \alpha_0$ and α_0 is the angle of incidence from the vacuum. It has been shown in Ref. [8] that the capacitor model represents an excellent approximation for nonrelativistic intensities. With these simplifications the corresponding Maxwell equations read ($E = E_x$)

$$\frac{\partial E}{\partial t} = i k_y c^2 B + \frac{e}{\varepsilon_0} n_e u - \frac{e}{\varepsilon_0} n_0 v_0, \quad (6)$$

$$\frac{\partial E}{\partial x} = \frac{e}{\varepsilon_0}(n_0 - n_e). \quad (7)$$

From these relations the total time derivative of E is calculated as

$$\frac{dE}{dt} = \frac{\partial E}{\partial t} + u \frac{\partial E}{\partial x} = i c^2 k_y B + \frac{e}{\varepsilon_0} n_0(x) \{u(x, t) - v_0(x, t)\}. \quad (8)$$

The equation of motion of the electrons is

$$\frac{du}{dt} = -\frac{s_e^2}{n_e}\frac{\partial n_e}{\partial x} - \frac{e}{m_e} E, \quad (9)$$

where $s_e = (\gamma_e k T_e/m_e)^{1/2}$ represents the electron sound speed. Taking the total time derivative of this equation with the help of (8), we get

$$\frac{d^2 u}{dt^2} + \frac{d}{dt} s_e^2 \frac{1}{n_e}\frac{\partial n_e}{\partial x} + \omega_p^2 \{u(x, t) - v_0(x, t)\} = -i \frac{e c^2}{m_e} k_y B(x, t). \quad (10)$$

This equation is valid for arbitrarily large amplitudes u and for non-stationary ion density profiles, too, the only limitation being imposed by wave breaking. The plasma frequency is determined by the ion density,

$$\omega_p^2 = e^2 n_0(x, t)/\varepsilon_0 m_e,$$

which has to be taken at the instantaneous position x of an oscillating electron. It is now convenient to pass from Eulerian coordinates (x, t) to Lagrangean ones (a, t); here a is the initial position $x(a, 0) = a$ of a volume element. It is assumed that in the very overdense region $(\omega_p \gg \omega)$ it holds that $n_e = n_0$ because B is zero there. In the equilibrium state the departure $n_e - n_0$ is easily determined from the electron momentum equation $\nabla p_e + n_e \pi + e n_e E_s = 0$ and (7) applied to the static electric field E_s. $-n_e \pi$ is the ponderomotive force density. For most relevant cases $n_e - n_0$ is negligible. In regions of $\nabla p_e + n_e \pi = 0$, $n_e = n_0$ is exactly fulfilled. Introducing the relative electron flow velocity $v_e(a, t) = u(a, t) - v_0(a, t)$, the actual positions of the ion and electron fluid elements, starting from the same initial position, are

$$x_0(a, t) = a + \int_0^t v_0(a, t)\,\mathrm{d}t,$$

$$x(a, t) = x_0(a, t) + \int_0^t v_e(a, t)\,\mathrm{d}t = x_0(a, t) + x_e(a, t).$$

By means of these relations (10) transforms into the equation

$$\frac{\partial^2 v_e}{\partial t^2} + \frac{s_e^2(a, 0)}{\gamma n_0^\gamma(a, 0)} \frac{\partial^2 n_e^\gamma}{\partial a\, \partial t} + \omega_p^2\{x(a, t), t\}(v_e + \Delta v_0) = -i \frac{e c^2}{m_e} k_y B - \frac{\partial^2 v_0}{\partial t^2}. \tag{10a}$$

Δv_0 is given by the difference $\Delta v_0 = v_0\{x(a, t), t\} - v_0(a, t)$. It is due to the fact that at time t the ion fluid element starting from point a is at position $x_0(a, t)$, whereas the corresponding electron fluid element has moved to position $x(a, t)$. Before wavebreaking this equation is exact; after it will change because the current in (6) has to be calculated differently.

In the following we treat the effect of plasma motion on wave breaking. Therefore, in order to simplify (10a) considerably, we assume a cold plasma, i. e. $s_e = 0$. The term containing ω_p^2 can be approximated as follows:

$$\omega_p^2\{x(a, t), t\}\{v_e(a, t) + \Delta v_0\} \cong \omega_p^2\{x_0(a, t), t\}\, v_e(a, t)$$

as long as the electrons do not oscillate over too large an inhomogeneity region. The exact term would generate all higher harmonics. But those are not resonantly driven and can therefore be neglected. In addition, $\partial v_0 / \partial t = 0$ can be assumed. We justify these assumptions in the last Section of the paper, of course. The final equation now reads

$$\frac{\partial^2}{\partial t^2} v_e(a, t) + \omega_p^2(x_0, t)\, v_e(a, t) = -i \frac{e c^2}{m_e} k_y B \tag{11}$$

with

$$\omega_p^2(x_0, t) = e^2 n_0(a, t) / \varepsilon_0 m_e,$$

where $n_0(a, t)$ means $n_0\{x_0(a, t), t\}$. The electrons oscillate around the ion volume element of density $n_0(a, t)$ (oscillation center approximation). Since the plasma is streaming, ω_p becomes time dependent: A volume element streaming through the resonance region starts with $\omega_p \gg \omega$ (ω frequency of the driver), reaches $\omega_p = \omega$ at the critical point and then changes to $\omega_p < \omega$ owing to the rarefaction of the plasma.

The phenomenon of hydrodynamic wave breaking takes place as soon as the transformation from Eulerian to Lagrangean variables becomes multivalued. It can be shown that the necessary and sufficient condition for this to occur is that the inequality

$$\frac{\partial}{\partial a} \int_0^t v_e(a, t)\,\mathrm{d}t \leqq -\frac{n_0(a, 0)}{n_0(a, t)} \tag{12}$$

is fulfilled [9]. This criterion is invariant with respect to a shift of the point $t = 0$. Introducing

$$v = v_e / A, \quad A = -i \frac{e c^2}{m_e} k_y \hat{B}$$

and assuming sinusoidal time variation of the driver, $B = \hat{B} e^{-i\omega t}$, we get the following equation of a driven harmonic oscillator with time varying eigenfrequency for the relative electron velocity:

$$\frac{\partial^2}{\partial t^2} v(a, t) + \omega_p^2(a, t)\, v(a, t) = e^{-i\omega t}. \tag{13}$$

It is not possible to solve this equation analytically for the general case. When $v(a, t)$ oscillates only a few times during resonance with its driver the maximum amplitude depends very much on the relative phase between $e^{-i\omega t}$ and $v(a, t)$. But this case is neither interesting in our context because v

will remain small. In laser plasmas, for instance, v oscillates many times when it is resonant, typically from 10^2 to 10^3 times. A WKB approximation is then appropriate. The solution of the homogeneous part of (13) is given by

$$v(a,t) = \frac{C+}{\omega_p^{1/2}} e^{i\varphi(a,t)} + \frac{C-}{\omega_p^{1/2}} e^{-i\varphi(a,t)},$$

$$\varphi(a,t) = \int_0^t \omega_p(a,t)\,dt, \tag{14}$$

and consequently a solution of the inhomogeneous equation is

$$v = \frac{i}{2\omega_p^{1/2}}\left\{e^{-i\varphi}\int_0^t \frac{e^{-i(\omega t-\varphi)}}{\omega_p^{1/2}}\,dt \right. \tag{15}$$

$$\left. - e^{i\varphi}\int_0^t \frac{e^{-i(\omega t+\varphi)}}{\omega_p^{1/2}}\,dt\right\} =: v_1 - v_2\,.$$

For the special case of $\omega_p = \omega_0 = \text{const}$ solution (14) reduces to the well-known solution for $\omega \neq \omega_0$,

$$v = e^{i\omega t}/(\omega_0^2 - \omega^2)\,,$$

and for the case of resonance ($\omega = \omega_0$) to

$$v = \frac{i}{2\omega_0} t e^{-i\omega_0 t}\left\{1 - \frac{i}{2\omega_0 t}\right\},$$

which is nearly the smooth solution (2) because of $2\omega t \gg 1$ in our case. In order to get the exact solution for this case also, the correct linear combination of (14) and (15) has to be taken. In the resonance region $|v|$ starts growing because the phase $\omega t - \varphi$ becomes stationary. Consequently, only v, of (15) is important and v_2 can be disregarded because $|v_2/v_1| \ll 1$ is valid in very good approximation in the resonance region. The approximate solution of (13) we use in the following is therefore

$$v = \frac{i}{2\omega_p^{1/2}} e^{-i\varphi}\int_0^t \frac{e^{-i(\omega t-\varphi)}}{\omega_p^{1/2}}\,dt\,. \tag{16}$$

It also shows the correct behaviour $v \to 0$ for a volume element when it is still far from the resonance point ($\omega_p \gg \omega$). Furthermore it is interesting to note that, in contrast to the warm plasma case, the oscillation frequency of a cold plasma slab changes following ω_p adiabatically.

3. A Stationary Self-Consistent Ion Density Profile

Integral (16) is simplest to evaluate for ω_p linear in time. In order to find reasonable behaviour of ω_p in space, we imagine that a stationary self-consistent density step due to rarefaction and the ponderomotive force $n_0\pi$ has already been established. In general, formulation of this problem would lead to a nonlinear integral equation. For ω_p linear in time the correct profile $n_0(x) = n_0(a,t)$ can be determined in the following way.

From $\omega_p = \omega(1 + f(a) - \alpha t)$ with $f(a)$ the spatial variation to be determined it follows that (index c indicates the resonance point)

$$n_0 = n_c(1 + f(a) - \alpha t)^2 \quad \text{and}$$

$$v_0 = \frac{v_c}{(1 + f(a) - \alpha t)^2}\,.$$

The position $x(a,t)$ of a volume element is given by

$$x(a,t) = a + \int_0^t v_0\,dt$$

$$= a + \frac{v_c}{\alpha}\left\{\frac{1}{1+f(a)-\alpha t} - \frac{1}{1+f(a)}\right\}$$

and consequently $v_0(x)$ depends on x:

$$v_0(x) = v_c\left\{\frac{\alpha}{v_c}(x-a) + \frac{1}{1+f(a)}\right\}^2\,.$$

v_0 is stationary only if

$$\frac{1}{1+f(a)} - \frac{\alpha a}{v_c} = \pm 1$$

holds. Taking the + sign, v_0 increases in the positive x-direction. With this choice we obtain the self-consistent profiles

$$\omega_p = \omega\left(\frac{1}{1+\frac{\alpha}{v_c}a} - \alpha t\right),$$

$$\varphi(a,t) = \omega t\left(\frac{1}{1+\frac{\alpha}{v_c}a} - \frac{\alpha}{2}t\right), \tag{17}$$

$$n_0(x) = \frac{n_c}{\left(1+\frac{\alpha}{v_c}x\right)^2} = \frac{n_c}{\left(1+\frac{\alpha}{v_c}a\right)^2}$$

$$= n_0(a,0)\,, \tag{18}$$

$$v_0(x) = v_c\left(1 + \frac{\alpha}{v_c}x\right)^2 = v_c\left(1 + \frac{\alpha}{v_c}a\right)^2 = v_0(a, 0). \tag{19}$$

The scale length of n_0 depends on α as follows:

$$L(x) = \left|\frac{1}{n_0}\frac{\partial n_0}{\partial x}\right|^{-1} = \frac{\alpha x + v_c}{2\alpha}, \quad L(0) = L_c = \frac{v_c}{2\alpha}. \tag{20}$$

With the same procedure self-consistent distributions for the case of constant accerleration $v_0 = f(a) + 2\beta_0 t$ can be found:

$$\omega_p = \omega \frac{v_c^{1/2}}{\{(v_c^2 + 4\beta_0 a)^{1/2} + 2\beta_0 t\}^{1/2}},$$

$$\varphi(a, t) = \frac{\omega v_c^{1/2}}{\beta_0}\{|(v_c^2 + 4\beta_0 a)^{1/2} + 2\beta_0 t|^{1/2} - |v_c^2 + 4\beta_0 a|^{1/4}\},$$

$$n_0(x) = \frac{n_c}{\left(1 + 4\frac{\beta_0}{v_c^2}x\right)^{1/2}},$$

$$v_0(x) = (v_c^2 + 4\beta_0 x)^{1/2}.$$

We evaluate (16) for the simpler case of ω_p taken from (17). With the following magnitudes

$$\tau = -\frac{a}{v_c + \alpha a}, \quad \eta = \left(\frac{\omega\alpha}{2}\right)^{1/2}(t - \tau),$$

$$dt = \left(\frac{2}{\omega\alpha}\right)^{1/2} d\eta, \ \eta_0 = \left(\frac{\omega\alpha}{2}\right)^{1/2}\tau, \quad \gamma = \left(\frac{2\omega}{\alpha}\right)^{1/2},$$

we get

$$v(a, t) = i\frac{e^{-i\omega\tau}}{(2\alpha)^{1/2}\omega^{3/2}} \cdot \frac{e^{i\eta(\eta-\gamma)}}{\left(1 - \frac{2\eta}{\gamma}\right)^{1/2}} \cdot \int_{-\eta_0}^{\eta} \frac{e^{-i\eta^2}}{\left(1 - \frac{2\eta}{\gamma}\right)^{1/2}} d\eta, \tag{21}$$

where τ is the time a volume element takes to travel from its initial position $x_0 = a < 0$ to the point of resonance $x_0(a, t) = 0$. Since γ is a large quantity,

$$\gamma = \left(\frac{2\omega}{\alpha}\right)^{1/2} = 2\left(\frac{\omega L_c}{v_c}\right)^{1/2} = 2\left(\frac{k_0 L_c}{\beta}\right)^{1/2}, \quad \beta = \frac{v_c}{c},$$

typically $\gamma \gtrsim 10^2$, the denominators $(1 - 2\eta/\gamma)^{1/2}$ are slowly varying quantities and are nearly unity around the resonance point $\eta = 0$. The main contribution to the amplitude $|v|$ thus comes from the integral around $\eta = 0$, where the phase is stationary:

$$|v| = \frac{1}{(2\alpha)^{1/2}\omega^{3/2}\left(1 - \frac{2\eta}{\gamma}\right)^{1/2}}\left|\int_{-\eta_0}^{\eta}\frac{e^{-i\eta^2}d\eta}{\left(1 - \frac{2\eta}{\gamma}\right)^{1/2}}\right| \cong \frac{1}{(2\alpha)^{1/2}\omega^{3/2}\left(1 - \frac{2\eta}{\gamma}\right)}\left|\int e^{-i\eta^2}d\eta\right|, \tag{22}$$

i.e. the time behaviour of the amplitude is mainly determined by the value of the Fresnel integral $\int e^{-i\eta^2}d\eta$. Its absolute value is visualized as the radius vector on the Cornu spiral [10] (see Figure 1). The whole expression for $|v|$ is represented in Fig. 2 as a function of ω_p. The slow increase of $|v|$ out of resonance ($\omega_p < \omega$) is due to the WKB term $(1 - 2\eta/\gamma)^{-1}$. The small ripples above resonance ($\omega_p > \omega$) in Fig. 2 can easily be explained: The calculation starts at a finite time instead of $-\infty$, which means that in Fig. 1 the origin of the vector lies in a point on the spiral curve and not at $\eta = -\infty$. The ratio $\varkappa$ of the rates with which the phases $\eta(\eta - \gamma)$ and η^2 vary in (21) is given by

$$\varkappa = \left|\frac{\gamma}{2\eta} - 1\right|,$$

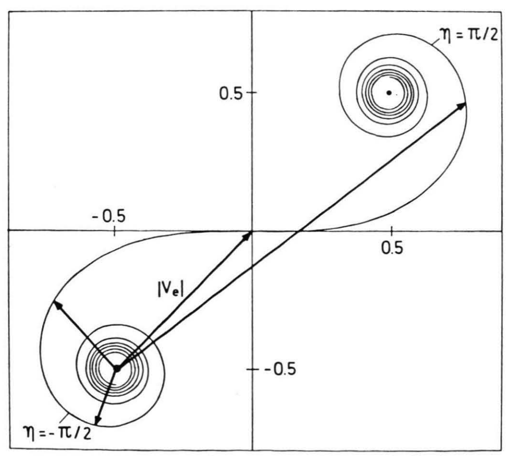

Fig. 1. The value of $\left|\int_{-\infty}^{\eta} e^{-i\eta^2}d\eta\right|$ is the length of the vector extending from the center of the lower Cornu spiral to a point of the double spiral determined by the parameter η, which is proportional to the length of the arc, i.e. time. Maximum growth occurs around the resonance $\eta = 0$. The maximum of $\left|\int_{-\infty}^{\eta} e^{-i\eta^2}d\eta\right| = 2.074$ is reached at $\eta = 1.53$ ($\pi/2 = 1.57$).

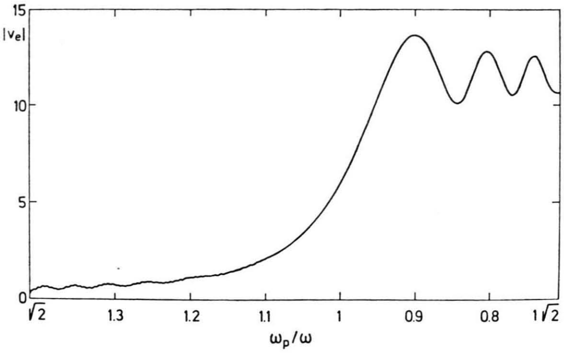

Fig. 2. Amplitude $|v|$ according to formula (22) as a function of ω_p varies from $\sqrt{2}\,\omega$ to $\omega/\sqrt{2}$ during 10^2 oscillations. Growth of $|v|$ occurs around the resonance from $1.1\,\omega$ to $0.9\,\omega$. Afterwards the oscillator gets out of phase. The modulations decrease as $t^{1/2}$ (or $\eta^{1/2}$).

which shows that as soon as $|\eta|$ is less than $\gamma/10$ the phase $\eta(\eta-\gamma)$ changes at least four times more rapidly than the exponent of the Fresnel integral. In the resonance region $|\eta| \lesssim \pi/2$ for the reasonable value $\gamma = 100\varkappa$ is at least 30 everywhere. It thus becomes apparent that it makes sense to define expression (22) as the usual amplitude of the oscillation.

The driving electric field $\hat{E}_d$ can be defined as $\hat{E}_d = c\,\hat{B}\sin\alpha_0$. In addition, by setting $v_d = -ieE_d/m_e\omega = -A/\omega^2$, which is a linearized solution of (13) in vacuum ($\omega_p = 0$), (22) is written as

$$\left|\frac{v_e}{v_d}\right| = \frac{\gamma}{2\left(1-\dfrac{2\eta}{\gamma}\right)}\left|\int_{-\eta_0}^{\eta} e^{-i\eta^2}\mathrm{d}\,\eta\right|, \tag{23}$$

$$\left|\frac{v_e}{v_c}\right| = \frac{e\hat{E}_d}{m_e c\,\omega}\cdot\frac{(k_0 L_c)^{1/2}}{\beta^{3/2}\left(1-\dfrac{2\eta}{\gamma}\right)}\left|\int_{-\eta_0}^{\eta} e^{-i\eta^2}\mathrm{d}\eta\right|.$$

The multiplication factor due to the electron response $|v_e/v_d|$ in the resonance region is highest at $\eta \simeq \pi/2$

$$\frac{v_e}{v_d} = 1.04\,\frac{\gamma}{1-\pi/\gamma}.$$

The width Δ over which energy is coupled from the driving field into the plasma oscillation is determined from the resonance width

$$\Delta\eta = \pi = (\omega\alpha/2)^{1/2}\Delta t$$

and $\Delta t = \int \mathrm{d}x/v(x)$. The calculation yields for Δ

$$\frac{1}{1-(\Delta/4L_c)^2}\cdot\frac{\Delta}{L_c} = 2\pi\left(\frac{\beta}{k_0 L_c}\right)^{1/2}. \tag{24}$$

4. Criterion of Wave Breaking in a Streaming Cold Plasma

In order to get the desired criterion for hydrodynamic wave breaking as a function of driver frequency ω, incident intensity I_{inc}, flow velocity v_c and scale length L_c, we have (I) to integrate $v_e(a,t)$, (II) to take the maximum of $|\partial/\partial a \int v_e \mathrm{d}t|$ and (III) to relate the driver field to I_{inc} and to the most favourable angle of incidence. In this way the lowest possible wave breaking threshold is determined.

Because of $\eta = (\omega\alpha/2)^{1/2}(t-\tau)$ and $\tau = -a/(v_c+\alpha a)$ it holds for v from (21)

$$\frac{\partial}{\partial a}\int_0^t v\,\mathrm{d}t = \left(\frac{2}{\omega\alpha}\right)^{1/2}\frac{\partial}{\partial a}\int_{-\eta_0}^{\eta} v\,\mathrm{d}\eta$$

$$= \left(\frac{2}{\omega\alpha}\right)^{1/2}\left\{-i\omega\frac{\partial\tau}{\partial a}\int_{-\eta_0}^{\eta} v\,\mathrm{d}\eta + v\frac{\partial\eta}{\partial a}\right\}.$$

One has $v\dfrac{\partial\eta_0}{\partial a} \simeq 0$ since v is zero at $-\eta_0 \ll 0$. We also obtain

$$\frac{\partial\tau}{\partial a} = -\frac{v_c}{(v_c+\alpha a)^2},$$

$$\frac{\partial\eta}{\partial a} = \frac{v_c}{(v_c+\alpha a)^2}\left(\frac{\omega\alpha}{2}\right)^{1/2},$$

and finally

$$\frac{\partial}{\partial a}\int_0^t v\,\mathrm{d}t = \frac{v_c}{(v_c+\alpha a)^2}\left\{v + i\gamma\int_{-\eta_0}^{\eta} v\,\mathrm{d}\eta\right\}. \tag{25}$$

We now evaluate $\int v\,\mathrm{d}\eta$. Integrating v by parts, we obtain the relation

$$\frac{\partial}{\partial\eta}\left\{\frac{v}{1-\dfrac{2\eta}{\gamma}} + \frac{e^{-i\omega t}}{\omega^2\left(1-\dfrac{2\eta}{\gamma}\right)^2}\right\} \tag{26}$$

$$= i\gamma v\left(1+\frac{3i}{\gamma^2\left(1-\dfrac{2\eta}{\gamma}\right)^2}\right) - \frac{4}{\gamma}\,\frac{e^{-i\omega t}}{\omega^2\left(1-\dfrac{2\eta}{\gamma}\right)^3}.$$

The RHS can be approximated by $i\gamma v$ to the high precision of $1/\gamma^2$ as long as η is not taken too far out in the underdense region (e.g. $\eta \leqq \gamma/3$). Keeping in mind that the lower limit of integration can be taken so far away from resonance (i.e. $\omega_p^2 \gg \omega^2$) that the contribution to the integral from $-\infty$ to $+\infty$ becomes arbitrarily small, one then obtains the following result:

$$v + i\gamma \int_{-\eta_0}^{\eta} v \, d\eta = \frac{e^{-i\omega t}}{\omega^2 \left(1 - \dfrac{2\eta}{\gamma}\right)^2} \left\{ \frac{1}{2} + i \left(1 - \frac{2\eta}{\gamma}\right)^{1/2} \gamma \, e^{i\eta^2} \int_{-\eta_0}^{\eta} \frac{e^{-i\eta^2}}{\left(1 - \dfrac{2\eta}{\gamma}\right)^{1/2}} d\eta \right\}. \tag{27}$$

From (12), (17), (25) and (27) and with $\omega(1 - 2\eta/\gamma) = \omega_p$ the following condition for wave breaking is now arrived at in a straightforward way:

$$-\frac{v_c}{(v_c + \alpha a)^2 \, \omega_p^2(a, t)} \, \Re e \left\{ e^{-i\omega t} A \left(\frac{1}{2} + i \left(1 - \frac{2\eta}{\gamma}\right)^{1/2} \eta \, e^{i\eta^2} \int_{-\eta_0}^{\eta} \frac{e^{-i\eta^2}}{(1 - 2\eta/\gamma)^{1/2}} d\eta \right) \right\} < -\frac{\omega_p^2(a, 0)}{\omega_p^2(a, t)} .$$

Since $\omega_p^2(a, 0) = \omega^2 v_c^2/(v_c + \alpha a)^2$ holds, inequality (28) explicitely shows that our wave breaking condition (12) does not depend on the initial position of a volume element as soon as it is chosen well above resonance, in accordance with physical intuition.

Criterion (28) can still be simplified by considering that ωt changes much more rapidly than the other quantities in the bracket. Therefore, instead of looking at $\Re e\{\ \}$ it is perfectly equivalent to consider the absolute value of this bracket. In addition, if we limit η to the still large interval $\eta \lesssim \gamma/10$, $(1 - 2\eta/\gamma)^{1/2}$ can be ignored and

$$\frac{|A|}{\omega^2 v_c} \left| \frac{1}{2} + i\eta \, e^{i\eta^2} \int_{-\infty}^{\eta} e^{-i\eta^2} d\eta \right| > 1 \tag{29}$$

or, equivalently,

$$\left| \frac{v_d}{v_c} \right| > \frac{1}{\left| \frac{1}{2} + i\eta \, e^{i\eta^2} \int_{-\infty}^{\eta} e^{-i\eta^2} d\eta \right|}$$

holds.

These formulae represent the desired, rigorous criterion for wave breaking in a cold inhomogeneous plasma which streams with velocity v_c at the resonance point. The absolute value of the Fresnel integral reaches its maximum around $\eta = \pi/2$. However, the absolute value in inequality (29) increases monotonically with η as can be seen from Figure 3. As a consequence, far out from resonance the threshold of wave breaking is drastically lowered. For a special set of parameters this behaviour is shown in Figure 4: While the first density peak is still moderate, the third spike has already broken. The physical reason for such behaviour is partly due to the wavelength decrease of the electrostatic oscillations in the lower density region.

We are interested here in the wave breaking threshold in the resonance region. We therefore have to evaluate inequality (29) at $\eta = \pi/2$. There are several reasons for this point. If we investigated breaking in the lower density region, besides temperature effects, damping would play a decisive role. Breaking there depends on how geometrical effects, damping and dispersion compete with each other. On the other hand, although there is damping, it is reasonable to assume that the oscillation amplitude will grow over the whole resonance width Δ because this is exactly the region where oscillations can be resonantly excited. In addition, break-

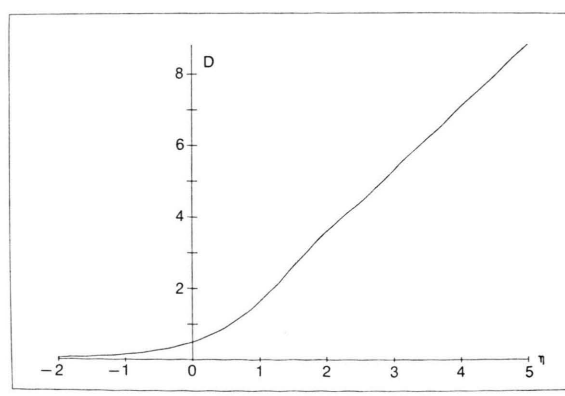

Fig. 3. The value of the denominator

$$D = \left| \tfrac{1}{2} + i\eta e^{i\eta^2} \int_{-\infty}^{\eta} e^{-i\eta^2} d\eta \right|$$

is a monotonically increasing function of η, i.e. time. At $\eta = \pi/2$ the value $D = 2.759$ is assumed.

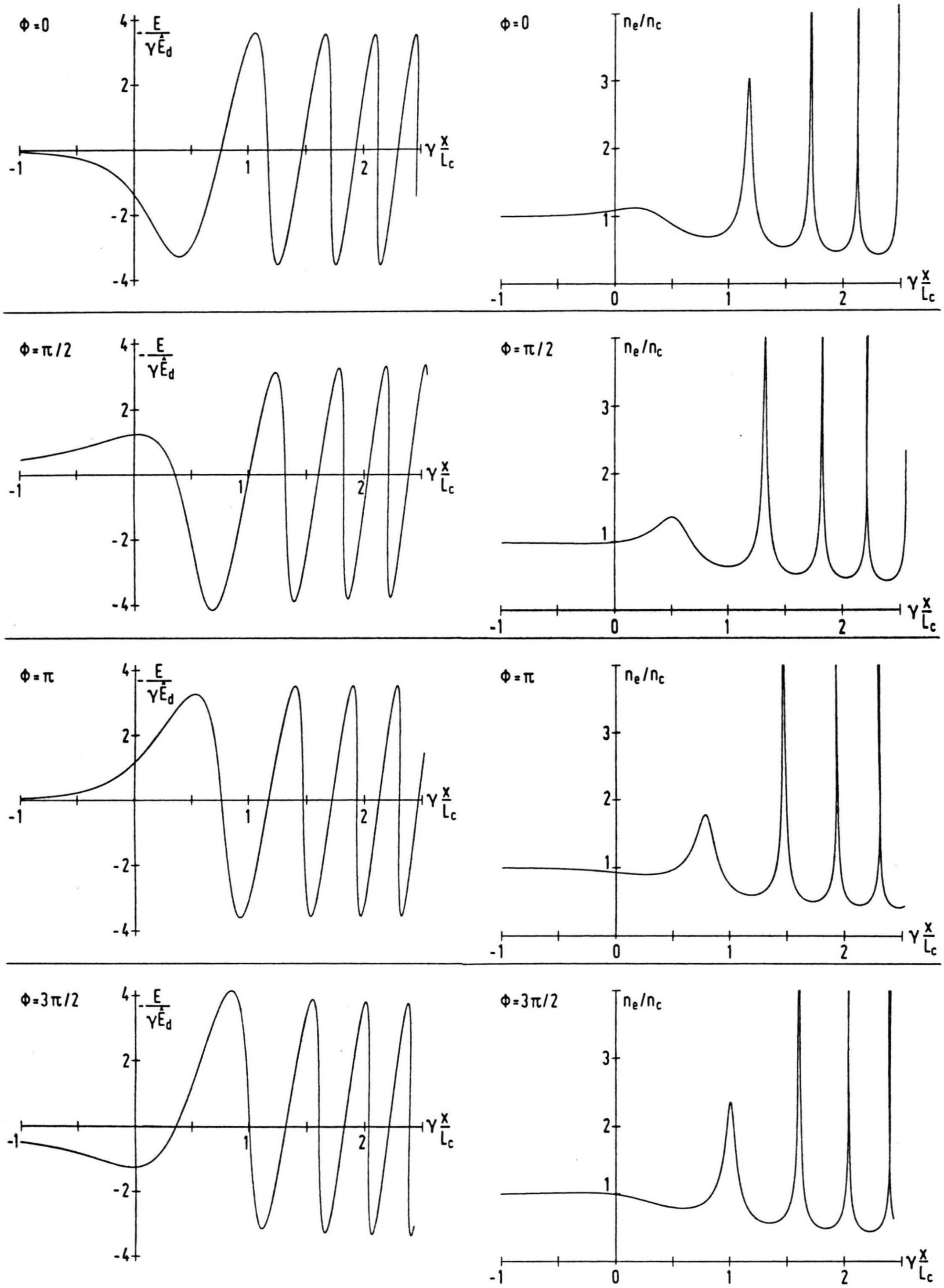

Fig. 4. LHS: Electrostatic wave around resonance point as a function of the Eulerian space coordinate at four different times. RHS: The corresponding electron density. $|v_d/v_c| = 0.16$ was chosen. Breaking occurs in the third density spike.

ing at the point $\eta=\pi/2$ can be neatly related to the simple model of [3] (see Section 5, b). At this point inequality (29) reads

$$\left|\frac{v_\mathrm{d}}{v_\mathrm{c}}\right| > \frac{1}{\left|\frac{1}{2}+i\frac{\pi}{2}e^{i(\pi^2/4)}\int\limits_{-\infty}^{\pi/2} e^{-i\eta^2}\,\mathrm{d}\eta\right|} = 0.3625 \tag{30}$$

and

$$\left|\frac{v_\mathrm{e}}{v_\mathrm{c}}\right| = \left|\frac{v_\mathrm{e}}{v_\mathrm{d}}\right|\left|\frac{v_\mathrm{d}}{v_\mathrm{c}}\right| = \frac{\gamma}{2\left(1-\frac{\pi}{\gamma}\right)}\left|\int\limits_{-\infty}^{\pi/2} e^{-i\eta^2}\,\mathrm{d}\eta\right|\left|\frac{v_\mathrm{d}}{v_\mathrm{c}}\right| > 0.374\,\frac{\gamma}{1-\frac{\pi}{\gamma}}\,.$$

With the help of (24) the latter inequality can be expressed as a condition for the oscillation amplitude $x_\mathrm{e}=i\,v_\mathrm{e}/\omega_\mathrm{p}$,

$$\left|\frac{x_\mathrm{e}}{\Delta}\right| = \frac{1}{2\pi[1-(\Delta/L_\mathrm{c})^2]\left(1-\frac{\pi}{\gamma}\right)^2}\left|\frac{v_\mathrm{d}}{v_\mathrm{c}}\right|\left|\int\limits_{-\infty}^{\pi/2} e^{-i\eta^2}\,\mathrm{d}\eta\right| > \frac{0.748}{2\pi(1-\pi/\gamma)^2[1-(\Delta/L_\mathrm{c})^2]} \cong 0.12$$

i.e. breaking in the resonance region occurs when the oscillation amplitude reaches 1/10 of the resonance width.

The minimum B-field amplitude for breaking at $\eta=\pi/2$ is

$$\hat{B}\sin\alpha_0 > 0.36\,m_\mathrm{e}\,\omega\,\beta/e\,. \tag{31}$$

$\hat{B}$ now has to be related to the intensity of the driving wave. For $k_0 L_\mathrm{c} \gtrsim 1$ the following formula for $\hat{B}$ is accurate enough [11]:

$$\hat{B}\sin\alpha_0 = \hat{B}_\mathrm{inc}(1-\sin^2\alpha_0)^{1/4}\left(\frac{\sigma}{\pi}\right)^{1/2}(k_0 L_\mathrm{c})^{-1/2} = 5.16\times 10^{-6}\left(\frac{\sigma I_\mathrm{inc}}{k_0 L_\mathrm{c}}\right)^{1/2}(1-\sin^2\alpha_0)^{1/4}\,, \tag{32}$$

where $[I_\mathrm{inc}]=\mathrm{W/cm^2}$. $\hat{B}$ [Tesla] is given by $\hat{B}=9.14\times 10^{-6}$. $I_\mathrm{inc}^{1/2}$ and σ is the fraction of the resonantly absorbed driver. At the optimum angle [11]

$$\sin\alpha_0 \cong 0.71/(k_0 L_\mathrm{c})^{1/3}$$

it holds that $x \cong 0.5$. By means of these relations one obtains from (31)

$$I_\mathrm{inc} > 3.2\times 10^{-13}\,\omega^2\beta^2 \cdot \frac{k_0 L_\mathrm{c}}{[1-0.5/(k_0 L_\mathrm{c})^{2/3}]^{1/2}}\,. \tag{33}$$

This is the minimum intensity needed for wave breaking under optimum conditions. Owing to profile steepening by ponderomotive force $k_0 L_\mathrm{c}$-values between 10 and 2 have been measured [12]. Values of $\beta \gtrsim 10^{-3}$ are reasonable. In the case of a neodymium laser with $\omega=1.778\times 10^{15}$/s we obtain the following threshold

$$I_\mathrm{inc,Nd} > 1\cdot 1\times 10^{13}\,\mathrm{W/cm^2},\quad \beta=10^{-3},\quad k_0 L_\mathrm{c}=10\,,$$

$$I_\mathrm{inc,Nd} > 2\cdot 2\times 10^{12}\,\mathrm{W/cm^2},\quad \beta=10^{-3},\quad k_0 L_\mathrm{c}=2\,.$$

The latter value represents a lower limit even under ideal conditions of optimum angle and no damping because $\sigma=0.5$ is valid for $k_0 L_\mathrm{c} \gtrsim 2\pi$. At lower values of $k_0 L_\mathrm{c}$, σ is also reduced. It is therefore convenient to keep σ as a parameter and to write the criterion in the form

$$I_\mathrm{inc} > 1.6\times 10^{-13}\,\frac{\omega^2\beta^2 k_0 L_\mathrm{c}}{\sigma[1-0.5/(k_0 L_\mathrm{c})^{2/3}]^{1/2}}\,[\mathrm{W/cm^2}]\,. \tag{34}$$

The angular dependence only enters through absorption fraction σ and the weak correction $[1-0.5/(k_0 L_\mathrm{c})^{2/3}]^{1/2}$. Without light pressure $k_0 L_\mathrm{c}$-values of several 10^2 would easily be reached and the intensity threshold would increase correspondingly. Formulae (1) to (28) are valid for profiles $k_0 L_\mathrm{c} \ll 1$ too. However, it generally becomes more difficult to relate $\hat{B}$ to $\hat{B}_\mathrm{inc}$ in this case (see [11], PLF 25).

5. Discussion

In this section we point out some additional physical aspects. Further we show that the approximations introduced in the former sections are consistent.

a) It is interesting to note that the functional dependence in formula (31) can be obtained by a simple consideration. According to (2) a volume element at resonance gains the oscillation amplitude $x_e(0, t) = -i(e/2m_e\omega)t\hat{E}_d e^{-i\omega t}$, whereas the amplitude of the adjacent element at position a is

$$x_e(a, t) = -i\frac{e}{m_e\omega v_c}\hat{E}_d e^{-i\omega(t+a/v_\varphi)}\cdot\left(t+\frac{a}{v_c}\right)$$

because until time t it has spent an a/v_c longer period in the resonance region. $v_\varphi \geqq c$ is the phase velocity of the driver. From this we obtain

$$\frac{\partial x_e}{\partial a} = -i\frac{e}{2m_e\omega v_c}\hat{E}_d e^{-i\omega(t+a/v_\varphi)},$$

for the derivative of the exponential is negligible. With the breaking condition $|\partial x_e/\partial a| > n_0(a, 0)/n_0(a, t) \simeq 1$ the inequality results

$$\hat{E}_d > 2m_e\omega v_c/e,$$

which agrees exactly with the condition that breaking occurs at $\eta = 0$ (see Figure 3).

This differs from the correct expression (31) by the factor 5.5, i.e. this formula would give a 30 times higher threshold intensity for wave breaking than eq. (31). It clearly shows that the phase difference of adjacent volume elements plays a decisive role in the pheomenon of wave breaking.

b) In [3] the breaking time of a plasma at rest was calculated to be (in our units and symbols)

$$t_b = 2\left(\frac{2L_c m_e}{c\hat{B}\sin\alpha_0}\right)^{1/2}.$$

Thereby the phase shift out of resonance was also taken into account. Therefore acceptable agreement with our criterion (31) should be obtained if t_b is equated to the time $t = \pi(2/\alpha\omega)^{1/2}$ which is the time for resonant excitation. In fact, we get in this way the breaking condition

$$\hat{B}\sin\alpha_0 > 2m_e\omega\beta/\pi^2 e$$

which differs from the correct criterion by a factor of 1.8 only.

c) Equation (13) was obtained by introducing the oscillation center approximation which consists in substituting $\omega_p^2(x(a, t), t)$ by $\omega_p^2(x_0(a, t), t)$ in (10a). A first order expansion of the exact term yields

$$\begin{aligned}
&\omega_p^2(x(a, t), t)(v_e + \Delta v_0) \\
&= \omega_p^2(x_0, t)\, v_e + \frac{\partial\omega_p^2}{\partial x_0} x_e v_e + \omega_p^2\frac{\partial v_0}{\partial x_0} x_e \\
&\simeq \omega_p^2(x_0, t)\left(1 - i\frac{\omega^2}{\omega_p^5} v_c \frac{\partial\omega_p^2}{\partial x_0}\right)v_e + \frac{i}{\omega_p}\frac{\partial\omega_p^2}{\partial x_0} v_e^2.
\end{aligned}$$

It shows that Δv_0 results in a small correction of $\omega_p^2(x_0, t)$ by an amount $\beta/(k_0 L_c)$, whereas the second term generates all higher harmonics of ω_p. A simple perturbation analysis of (10a) yields the amplitude ratios

$$\frac{v_2}{v_e} = \frac{i}{6\omega_p^3}\frac{\partial\omega_p^2}{\partial x_0} v_e \simeq \frac{1}{6}\frac{x_e}{L_c},$$

$$\frac{v_3}{v_2} = \frac{3}{32}\frac{i}{\omega_p^3}\frac{\partial\omega_p^2}{\partial x_0} v_e \simeq \frac{3}{32}\frac{x_e}{L_c}. \qquad (35)$$

When breaking occurs in the resonance region $|x_e| = 0.12$ holds. On the other hand, formula (24) tells us that for $(\beta/k_0 L_c)^{1/2} < 1/10$, Δ is by at least a factor of 2 smaller than L_c, so that the oscillation center approximation is well justified.

d) Solution (16) holds as long as $\partial\omega_p/\partial t$ and $\partial^2\omega_p/\partial t^2$ can be neglected, i.e. as long as the oscillator undergoes enough oscillations (more than 5 for example) in the resonance region. The number n of oscillations there is

$$n \simeq \frac{\omega}{2\pi}\frac{\Delta}{v_c} = (k_0 L_c/\beta)^{1/2} = \frac{\gamma}{2}.$$

So, $\gamma/2$ expresses the resonant number of oscillations. It turns out again that $(\beta/k_0 L_c)^{1/2} < 1/10$ is a sufficient condition for the validity of solution (16), too. For $\beta \ll 1$ all formulae derived in the forgoing sections are correct for very steep density gradients as soon as the denominator $(1 - 2\eta/\gamma)^{1/2}$ is included in the Fresnel integral.

e) The density $n_e(a, t)$ is easily calculated from the mass conservation equation

$$n_e(a, t) = \frac{n_0(a, 0)}{\dfrac{\partial x_0}{\partial a} + \Re e\left(\dfrac{\partial x_e}{\partial a}\right)} \qquad (36)$$

$$= \frac{n_0(a, 0)}{1 + \dfrac{\partial}{\partial a}\int v_0\,dt + \dfrac{\partial}{\partial a} Re(A\int v\,dt)}.$$

From our special profile from Sect. 3 we calculate

$$x_0(a,t) = a + \frac{v_c}{\alpha}\left\{\frac{1}{1+\frac{\alpha}{v_c}a-\alpha t} - \frac{1}{1+\frac{\alpha}{v_c}a}\right\},$$

and

$$\partial x_0/\partial a = 1 + n_c(n_0(a,t) - n_0(a,0))/n_0(a,0) \cdot n_0(a,t).$$

$x_e(a, t)$ and $\partial x_e/\partial a$ are given by (25) and (27). The electric field is calculated best from (9):

$$E(a,t) = -\frac{m_e}{e}\left\{\frac{\partial v_0}{\partial t} + \Re e\frac{\partial v_e}{\partial t}\right\}$$

$$\cong \frac{m_e}{e}\omega\,|A|\left\{\frac{1}{1+a/2L_c} - \frac{v_c t}{2L_c}\right\}\Im m\, v(a,t).$$

In Fig. 4 the electric field and density distributions are shown as functions of the space coordinate for the parameter $|v_d/v_c| = 0.16$. According to (29) and Fig. 3 this leads to wave breaking at $\eta \cong 3.5$.

f) In the case of laser produced plasmas a linear treatment of resonance absorption in a warm plasma shows that hydrodynamic flow becomes important in the resonance region as soon as the following condition is fulfilled [13]

$$\frac{v_c}{s_e}(k_0 L_c/\beta)^{1/3}\xi \gtrsim 1, \quad (37)$$

where

$$\xi = (k_0^2/\beta^2 L_c)^{1/3}\left(x + \frac{v_c^2}{s_e^2}L_c\right)$$

is the dimensionless variable of the related inhomogeneous Stokes equation in which flow is included,

$$\frac{\partial^2 w}{\partial \xi^2} + \xi w = e^{-i\omega t - i(v_c/s_e)(k_0 L_c/\beta)^{1/3}\xi}$$

w oscill. velocity.

At the end of resonance this variable assumes the value $\xi = 1.85$.

We conclude that for a streaming cold plasma a consistent analytical treatment of hydrodynamic wave breaking can be given.

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